6th sem
(General) Transportation problem

Mathematical model of transportation problem

Let in a transportation problem, there are m origins $O_1, O_2, \cdots O_m$ with available quantities of $a_1, a_2, \cdots a_m$ and n destinations $D_1, D_2, \cdots D_n$ with demands $b_1, b_2, \cdots b_n$. Let c_{ij} $(i = 1, 2, \cdots m;\ j = 1, 2, \cdots n)$ denotes the transportation cost of one unit of the commodity to transfer from the i-th origin to the j-th destination. Our problem is to determine x_{ij}, the amount of units to be transferred from the i-th origin to the j-th destination, so that the total transportation cost $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$ be minimum. If it is also assumed that total availability equal to the total demand i,e, $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ and this type of T.P is known as balanced T.P.

This problem can be put in a tabular form as shown below.

Destinations

origins	D_1	D_2	D_3	$\cdots$	D_n	availability
O_1	$\boxed{x_{11}}$ c_{11}	$\boxed{x_{12}}$ c_{12}	$\boxed{x_{13}}$ c_{13}	$\cdots$	$\boxed{x_{1n}}$ c_{1n}	a_1
O_2	$\boxed{x_{21}}$ c_{21}	$\boxed{x_{22}}$ c_{22}	$\boxed{x_{23}}$ c_{23}	$\cdots$	$\boxed{x_{2n}}$ c_{2n}	a_2
$\vdots$	-	-				$\vdots$
O_m	$\boxed{x_{m1}}$ c_{m1}	$\boxed{x_{m2}}$ c_{m2}	$\boxed{x_{m3}}$ c_{m3}	$\cdots$	$\boxed{x_{mn}}$ c_{mn}	a_m
	b_1	b_2	b_3	$\cdots$	b_n	

Demand

Thus a transportation problem (T.P) can be mathematically put as a linear programming as below —

$$\left.\begin{aligned} &\text{minimize } z = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} c_{ij} \\ &\text{subject to } \sum_{j=1}^{n} x_{ij} = a_i \; ; \; i = 1, 2, \ldots m \\ &\qquad\qquad \sum_{i=1}^{m} x_{ij} = b_j \; ; \; j = 1, 2, \ldots n \\ &\qquad x_{ij} \geq 0 \text{ for all } i \text{ and } j. \end{aligned}\right\} \quad (1)$$

Q. Write down the following T.P into a L.P.P form.

	D_1	D_2	a_i
S_1	3	2	10
S_2	1	4	8
b_j	12	6	

Solution:- Reduced to L.P.P form we get

$$\min z = \sum_{i=1}^{2}\sum_{j=1}^{2} c_{ij} x_{ij}$$

Subject to $\displaystyle\sum_{i=1}^{2} x_{ij} = b_j$

$$\sum_{j=1}^{2} x_{ij} = a_i$$

(where c_{ij} be transportation cost of one unit of commodity transfered from i-th origin to j-th destination; x_{ij} be the amount of units to be transfered from i-th origin to j-th destination,

a_i be availability and b_j be demand)

now $\min z = 3x_{11} + 2x_{12} + x_{21} + 4x_{22}$

Subject to
$$3x_{11} + 2x_{12} = 10$$
$$x_{21} + 4x_{22} = 8$$
$$3x_{11} + x_{21} = 12$$
$$2x_{12} + 4x_{22} = 6$$

(Where $x_{11}, x_{12}, x_{21}, x_{22} \geq 0$)

H.W write down the following T.P into a L.P.P form

	D_1	D_2	D_3	a_i
O_1	5	1	8	12
O_2	2	4	0	14
O_3	3	6	7	4
b_j	9	10	11	

Q. What is balanced transportation problem?

Soln:- Let x_{ij} be the elements transported from i-th origin to j-th destination. Let c_{ij} be transported cost per unit commodity from i-th origin to j-th destination. ($i = 1, 2, \ldots m$; $j = 1, 2, \ldots n$)

Let a_i be availability at i-th origin and b_j be demand at j-th destination. Now the transportation problem is said to be balanced problem if $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ holds.

8. Write down Linear programming problem associated with a Transportation problem.

Solⁿ:- (1) is the required L.P problem associated with a T.P.

Some basic theorems on transportation problem

(1) The number of basic variables in a balanced transportation problem is at most $(m+n-1)$. Where T.P has m origins and n destinations.

Proof:- In a transportation problem (1) with m origins and n destinations, total number of constraints are $(m+n)$ and they are given by

$$\sum_{j=1}^{n} x_{ij} = a_i \quad (i=1,2,\cdots m) \quad \text{— (2)}$$

and $$\sum_{i=1}^{m} x_{ij} = b_j \quad (j=1,2,\cdots n) \quad \text{— (3)}$$

again since this is a balanced T.P so

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

now we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \quad (\text{using (2)}) \quad \text{— (4)}$$

With the help of this we shall show that one of the $(m+n)$ constraints is redundant i,e, one constraint can be obtained from the remaining constraints.

Summing the first $(n-1)$ constraints of (3) we get

$$\sum_{j=1}^{(n-1)} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{(n-1)} b_j \quad \text{———} (5)$$

substracting (5) from (4) we get

$$\sum_{i=m}^{} \left(\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{(n-1)} x_{ij} \right) = \sum_{j=1}^{n} b_j - \sum_{j=1}^{(n-1)} b_j$$

$$\Rightarrow \sum_{i=1}^{m} x_{in} = b_n$$

Which is the n-th constraint of (3). Thus the number of basic variables of a T.P is at most $(m+n-1)$.

Note

(1) From this theorem it may be concluded that any non degenerate basic feasible solution of a T.P will consist at most $(m+n-1)$ positive variables and the rest being zero. Also, by Fundamental theorem of L.P.P, one of these basi feasible solutions must be optimal.

(2)** When the number of positive variables in a basic feasible solution of a ~~bounded~~ balance T.P is exactly $(m+n-1)$, the solution is said to be non degenerate and when the number of positive variables in any B.F.S (Basic feasible solution) of a balanced T.P is less than $(m+n-1)$

the solution is said to be degenerate.

Q. What do you mean by degeneracy in T.P.

Ans from note - 2

THM 2 A necessary and sufficient condition (NASC) for the existence of a feasible solution to a T.P is $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

proof :-

Let the T.P 1 has a feasible solⁿ. Then we get

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i \text{ and}$$

$$\sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j$$

and thus from above two we get

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

The condition is sufficient

Let $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = \lambda$ (say)

we shall prove that $x_{ij} = \frac{a_i b_j}{\lambda}$ for all i and j, will be a feasible solution of the problem 1.

now $\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \frac{a_i b_j}{\lambda} = \frac{a_i}{\lambda}\sum_{j=1}^{n} b_j = \frac{a_i}{\lambda}\times\lambda = a_i$

$\therefore \sum_{j=1}^{n} x_{ij} = a_i \quad (i = 1, 2, --- m)$

Now $\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} \frac{a_i b_j}{d} = \frac{b_j}{d} \sum_{i=1}^{m} a_i = \frac{b_j}{d} \times d = b_j$

$\therefore \sum_{i=1}^{m} x_{ij} = b_j \quad (j = 1, 2, \ldots n)$

Which are the constraints of the T.P. Also $x_{ij} \geq 0$ since $a_i > 0$ and $b_j > 0$ for all i and j.

Thus x_{ij} is a feasible solution of the problem.

THM (3) The solution of a T.P in never unbounded.

Proof:- Let us consider the transportation problem (1). It is clear that each variable x_{ij} appears exactly twice in the constraints, appearing once in each of the set of constraints

$\sum_{i=1}^{m} x_{ij} = b_j$ and $\sum_{j=1}^{n} x_{ij} = a_i$; $(i = 1, 2, \ldots m$
$j = 1, 2, \ldots n)$

Also the co-efficients of each variable in any position is $(+1)$.

Since, $x_{ij} \geq 0$, $\sum_{i=1}^{m} x_{ij} = b_j$ and $\sum_{j=1}^{n} x_{ij} = a_i$ for $i = 1, 2, \ldots m$ and $j = 1, 2, \ldots n$

So, $0 \leq x_{ij} \leq \max(a_i, b_j)$

Since a_i, b_j are given finite quantites, so all x_{ij} are bounded.

Therefore, solution of T.P is never bounded.